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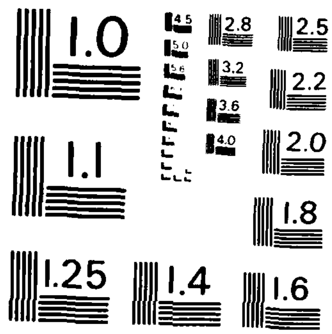

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COMPUTER SCIENCES DEPARTMENT

University of Wisconsin-Madison

ON AN ESTIMATE FOR

THE THREE-GRID MGR MULTIGRID METHOD

by

Seymour V. Parter

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ON AN ESTIMATE FOR

THE THREE-GRID MGR MULTIGRID METHOD*

by

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ABSTRACT

The MGR[ν] algorithm of Ries, Trottenberg and Winter, Algorithm 2.1 of Braess and Algorithm 4.1 of Verfürth are all algorithms for the numerical solution of the discrete Poisson equation based on red-black Gauss-Seidel smoothing iterations. In this work we consider the extension of the MGR[0] method to ~~the~~ general diffusion equation $-\nabla \cdot p\nabla u = f$. In particular, for the three grid scheme we extend an interesting and important result of Ries, Trottenberg and Winter whose results are based on Fourier analysis and hence intrinsically limited to the case where Ω is a rectangle. Let Ω be a general polygonal domain whose sides have slope ± 1, 0 and ∞. Let ε^0 be the error before a single multigrid cycle and let ε^1 be the error after this cycle. Then $\|\varepsilon^1\|_{L_h} \leq \frac{1}{2}(1+Kh)\|\varepsilon^0\|_{L_h}$ where $\| \ \|_{L_h}$ denotes the energy or operator norm. When $p(x,y) \equiv$ constant, then $K \equiv 0$.

1. Introduction

The MGR[ν] multigrid algorithms of Ries, Trottenberg and Winter [4], the Algorithm 2.1 of Braess [1], [2] and Algorithm 4.1 of Verfürth [5] are all algorithms for the numerical solution of the discrete Poisson equation (the usual 5-point difference equations with $\Delta x = \Delta y = h$) based on red-black Gauss-Seidel smoothing iterations. The analysis of [4] is based on Fourier Analysis and is restricted to the case where the basic domain Ω is a square. The analysis of [1], [2] and [5] is for a bounded polygonal domain Ω whose sides have slope $\pm 1,0$ and ∞ and is based on certain energy estimates and a particular interpretation of the matrix equations. While this is not explicitly stated, this interpretation can be viewed as a particular choice of I_H^h, I_h^H, I_{2h}^H, I_H^{2h} etc, the operators which carry on the communication between the grids.

Recently, Kamowitz and Parter [3] considered a generalization of the algorithms of Ries, Trottenberg and Winter and Braess. They consider the general diffusion equation

$$(1.1) \qquad \begin{aligned} \nabla \circ p(x,y)\nabla u &= f \quad \text{in} \quad \Omega , \\ u &= 0 \quad \text{on} \quad \partial\Omega \\ p(x,y) &\geq p_0 > 0 , \end{aligned}$$

in general domains Ω. Using a different choice of I_H^h, I_h^H than Braess, i.e. imagining a different interpolation structure in the space S_H, they employ other "Energy Estimates" to obtain the basic estimate - for a two grid scheme: let ε^0 denote the error before a single multigrid cycle and let ε^1 denote the error after that complete multigrid cycle, then

$$(1.2)\qquad \|\varepsilon^1\|_{L_h} \le \frac{1}{2}(1+Kh)\|\varepsilon^0\|_{L_h}$$

where the constant K depends only on p_0 and $\|\nabla p\|_\infty$, the ∞ norm of the gradient of $p(x,y)$ and $\|\ \|_{L_h}$ denotes the operator or energy norm. However, it is important to remark that despite the different interpretation of the problem, in the case of constant diffusion coefficient $p(x,y) \equiv 1$ we are dealing with exactly the same problem and the same iterative method. The estimate (1.2) is thus a generalization of the estimates

$$(1.3)\qquad \rho(MG) \le \frac{1}{2}, \quad \rho(MGR[0]) = \frac{1}{2}$$

of [1] and [4].

Another remarkable estimate of Ries, Trottenberg and Winter [4] is the fact that, in the case of Poisson equation in the square, if a third grid is introduced and one uses the MGR[0] method one obtains

$$(1.4)\qquad \rho(MGR[0],\ 3\ grid) = \frac{1}{2}.$$

In this report we obtain this estimate in the form (1.2) for the general diffusion equation (1.1) in bounded polygonal domains Ω whose sides have slope ± 1, 0 or ∞. We also require that the corners of Ω belong to the coarsest mesh. The constant K is a constant depending only on p_0, and the ∞ norm of the first and second derivatives of $p(x,y)$. Moreoever, if $p(x,y) \equiv$ const. then $K = 0$. In general, throughout this paper K will denote such a constant.

In section 2 we formulate the problem and the basic three-grid multigrid iteration. In particular we introduce the coarse grid operators L_H, $\hat{L}_H$, L_{2h}, $\hat{L}_{2h}$. In section 3 we develope more notation and recall some basic estimates of [3]. In this section the reader is introduced to a number of additional difference operators $L_H^{(1)}$, $\tilde{L}_H^{(1)}$, $L_{2h}^{(1)}$, $\tilde{L}_{2h}^{(1)}$, Q_x, M_x, $\bar{L}_x$. This plethora of operators gets a bit confusing. However if one first concentrates on the case $p(x,y) \equiv 1$ (i.e., the Poisson equation) the situation simplifies. In this case $L_H = L_H^{(1)}$, $L_{2h} = L_{2h}^{(1)}$ and (we always have) $\hat{L}_H = \frac{1}{2}L_H^{(1)} + \frac{1}{2}\tilde{L}_H^{(1)}$, $\hat{L}_{2h} = \frac{1}{2}L_{2h}^{(1)} + \frac{1}{2}\tilde{L}_{2h}^{(1)}$. Moreoever, in this case

$$\bar{L}_x = Q_x = L_{2h} = \tilde{L}_H^{(1)}\Big|_{\Omega_{2h}} ,$$

[Ω_{2h} is the coarsest grid] and

$$M_x = \tilde{L}_H^{(1)}\Big|_{\Omega_H/\Omega_{2h}}$$

[Ω_H is the intermediate grid]. Another observation which should be useful is the fact that, in this case $\tilde{L}_H^{(1)}$ is the same difference operator as L_{2h} <u>except</u> for points in Ω_H which are next to the boundary. Moreover, these exceptional points are in Ω_H/Ω_{2h} <u>not</u> in Ω_{2h}. This perturbation of $\tilde{L}_H^{(1)}$ causes a technical difficulty in the proof of lemma 5.2 <u>even</u> in this simplest case. In all cases the introduction of the variable diffusion coefficient $p(x,y)$ introduces perturbation of the basic operators. However, the essence of the proof of the main result [Theorem 5.1 or the estimate (1.2)] is contained in the constant coefficient case. The analysis of the algorithm is given in two parts, sections 4 and 5.

2. The Problem

Given a (small) value $h > 0$ let $\{(x_k,y_j) = (kh,jh);\ k,j = 0,\pm 1,\pm 2,\dots\}$ be the associated mesh points in the $x-y$ plane. Let

(2.1) $$R_0 := \{(x_k,y_j);\ k+j \equiv 1 (\mathrm{mod}\ 2)\}$$

(2.2) $$R_B := \{(x_k,y_j);\ k \equiv j \equiv 0\ (\mathrm{mod}\ 2)\}$$

(2.3) $$R_G := \{(x_k,y_j);\ k \equiv j \equiv 1\ (\mathrm{mod}\ 2)\}\ .$$

Let Ω be a bounded polygonal domain in the plane whose sides have slope $\pm 1, 0$, or ∞, and every corner point (x,y) of $\partial\Omega$ belongs to R_B. Define

(2.4a) $$\Omega_h = (R_0 \cup R_B \cup R_G) \cap \Omega$$

(2.4b) $$\partial\Omega_h = (R_0 \cup R_B \cup R_G) \cap \partial\Omega$$

(2.5a) $$\Omega_H = (R_B \cup R_G) \cap \Omega$$

(2.5b) $$\partial\Omega_H = (R_B \cup R_G) \cap \partial\Omega$$

(2.6a) $$\Omega_{2h} = R_B \cap \Omega$$

(2.6b) $$\partial\Omega_{2h} = R_B \cap \partial\Omega\ .$$

For any function $F(x,y)$ defined on $\overline{\Omega}$ we write:

(2.7a) $$F_{k,j} = F(x_k, y_j) ,$$

(2.7b) $$F_{k+\frac{1}{2},j} = F((k+\tfrac{1}{2})h, y_j) ,$$

(2.7c) $$F_{k,j+\frac{1}{2}} = F(x_k, (j+\tfrac{1}{2})h) .$$

The algebraic problem to be solved is: Find a mesh function $U = \{U_{kj}\}$ defined on $\Omega_h \cup \partial\Omega_h$ which satisfies

(2.8a) $$[L_h U]_{kj} = F_{kj} , \quad (x_k, y_j) \in \Omega_h$$

(2.8b) $$U_{kj} = 0 , \quad (x_k, y_j) \in \partial\Omega_h$$

where

(2.8c) $$[L_h U]_{kj} = \frac{1}{h^2}\{p_{k-\frac{1}{2},j}[U_{k,j} - U_{k-1,j}] - p_{k+\frac{1}{2},j}[U_{k+1,j} - U_{k,j}]\} + \frac{1}{h^2}\{p_{k,j-\frac{1}{2}}[U_{k,j} - U_{k,j-1}] - p_{k,j+\frac{1}{2}}[U_{k,j+1} - U_{k,j}]\} .$$

We turn to solution of these linear algebraic equations by a three-grid method.

Let S_h, S_H, S_{2h} be the linear spaces of mesh functions defined on $\Omega_h \cup \partial\Omega_h$, $\Omega_H \cup \partial\Omega_H$ and $\Omega_{2H} \cup \partial\Omega_{2h}$ respectively which vanish on the respective boundaries $\partial\Omega_h$, $\partial\Omega_H$, $\partial\Omega_{2h}$. We set up communication between these spaces. Specifically we define the linear interpolation and projection operators I_H^h, I_{2h}^H, I_h^H, I_H^{2h} as follows. The interpolation operator I_H^h (see the definition of I_E^h of [3]) is given by

(2.9a) $$I_H^h\colon S_H \to S_h$$

where

(2.9b) $$[I_H^h U]_{kj} = U_{kj}, \quad \text{if } (x_k, y_j) \in \Omega_H \cup \partial\Omega_H$$

and, if $(x_k, y_j) \in \Omega_h/\Omega_H$, then

(2.9c) $$[I_H^h U]_{kj} = \frac{1}{c_{kj}} \{p_{k-\frac{1}{2},j} U_{k-1,j} + p_{k+\frac{1}{2},j} U_{k+1,j} + p_{k,j-\frac{1}{2}} U_{k,j-1} + p_{k,j+\frac{1}{2}} U_{k,j+1}\}$$

where

(2.9d) $$c_{kj} = \{p_{k+\frac{1}{2},j} + p_{k-\frac{1}{2},j} + p_{k,j-\frac{1}{2}} + p_{k,j+\frac{1}{2}}\} .$$

Of course, if $(x_k, y_j) \in \partial\Omega_h/\partial\Omega_H$ then

(2.9e) $$[I_H^h U]_{kj} = 0 .$$

The projection operator I_h^H is defined by

(2.10) $$I_h^H = \tfrac{1}{2}(I_H^h)^T .$$

<u>Remark</u>: The factor $\frac{1}{2}$ in (2.10) is included merely to keep the method consistent with the MGR[ν] methods of [4].

The interpolation operator I_{2h}^H is defined in a similar manner by

(2.11a) $$I_{2h}^H\colon \; S_{2h} \to S_H$$

with

$$(2.11b)\qquad [I_{2h}^{H}U]_{kj} = U_{kj}\ , \quad \text{if} \quad (x_k,y_j) \in \Omega_{2h} \cup \partial\Omega_{2h}\ ,$$

and, if $(x_k,y_j) \in \Omega_H/\Omega_{2h} \cdots$ then

$$(2.11c)\qquad [I_{2h}^{H}U]_{kj} = \frac{1}{\bar{c}_{kj}}\{p_{k+\frac{1}{2},j+\frac{1}{2}}U_{k+1,j+1} + p_{k+\frac{1}{2},j-\frac{1}{2}}U_{k+1,j-1} + + p_{k-\frac{1}{2},j+\frac{1}{2}}U_{k-1,j+1} + p_{k-\frac{1}{2},j-\frac{1}{2}}U_{k-1,j-1}\}$$

where

$$(2.11d)\qquad \bar{c}_{kj} = \{p_{k+\frac{1}{2},j+\frac{1}{2}} + p_{k+\frac{1}{2},j-\frac{1}{2}} + p_{k-\frac{1}{2},j+\frac{1}{2}} + p_{k-\frac{1}{2},j-\frac{1}{2}}\}$$

and, if $(x_k,y_j) \in \partial\Omega_H/\partial\Omega_{2h}$, then

$$(2.11e)\qquad [I_{2h}^{H}U]_{kj} = 0\ .$$

The projection operator I_H^{2h} is given by

$$(2.12)\qquad I_H^{2h} = \frac{1}{2}(I_{2h}^{H})^T\ .$$

Finally we define the "coarse grid" operators L_H, L_{2h}. These are

$$(2.13a)\qquad L_H\colon S_H \to S_H$$

where, if $(x_k,y_j) \in \Omega_H$

$$(2.13b)\qquad [L_HU]_{kj} = \frac{1}{2h^2}\{\bar{c}_{k,j}U_{k,j} - p_{k+\frac{1}{2},j+\frac{1}{2}}U_{k+1,j+1} - p_{k+\frac{1}{2},j-\frac{1}{2}}U_{k+1,j-1} - p_{k-\frac{1}{2},j+\frac{1}{2}}U_{k-1,j+1} - p_{k-\frac{1}{2},j-\frac{1}{2}}U_{k-1,j-1}\}$$

and

$$L_{2h}: S_{2h} \to S_{2h} \tag{2.14a}$$

where, if $(x_h, y_j) \in \Omega_{2h}$ then

$$[L_{2h}U]_{kj} = \frac{1}{4h^2}\{p_{k-1,j}[U_{k,j} - U_{k-2,j}] - p_{k+1,j}[U_{k+2,j} - U_{k,j}]\} + \frac{1}{4h^2}\{p_{k,j-1}[U_{k,j} - U_{k,j-2}] - p_{k,j+1}[U_{k,j+2} - U_{k,j}]\} . \tag{2.14b}$$

We are now ready to describe the three grid methods. Let B_h be a non-singular linear operator defined on S_h

$$B_h: S_h \to S_h . \tag{2.15}$$

Let the smoothing operator G_h be defined by

$$G_h = I_h - B_h^{-1}L_h \tag{2.16a}$$

and assume that

$$\frac{\langle L_h G_h u, G_h u\rangle}{\langle L_h u, u\rangle} \leq 1 , \quad \forall u \in S_h, \quad u \neq 0 , \tag{2.16b}$$

<u>Algorithm</u>

<u>Step 1</u>: Given $u^0 \in S_h$, form

$$\tilde{u} = G_h u^0 + B_h^{-1} F . \tag{2.17}$$

Step 2: Perform one odd relaxation step. That is, construct $\hat{u}$ via

(2.18a) $$\hat{u}_{kj} = \tilde{u}_{kj}\,, \quad (x_k,y_j) \in \Omega_H$$

(2.18b) $$[L_h\hat{u}]_{kj} = f_{kj}\,, \quad (x_k,y_j) \in \Omega_h/\Omega_H$$

$$\hat{u}_{kj} = 0\,, \quad (x_k,y_j) \in \partial\Omega_H\,.$$

Step 3: Set $r = f - L_h\hat{u}\,, \quad r_H = I_h^H r\,.$

Step 4: Let $\hat{\psi}$ be obtained as follows.

(2.19a) $$\hat{\psi}_{ij} = 0\,, \quad (x_i,y_j) \in \Omega_{2h}$$

(2.19b) $$[L_H\hat{\psi}]_{ij} = r_H\,, \quad (x_i,y_j) \in \Omega_H/\Omega_{2h}\,.$$

Step 5: Set $\tilde{r}_H = r_H - L_H\hat{\psi}\,, \quad r_{2h} = I_H^{2h}\tilde{r}_H\,.$

Step 6: Solve

$$L_{2h}\phi = r_{2h}$$

Step 7: Set $u^1 = \hat{u} + I_H^h[\hat{\psi} + I_{2h}^H\phi]\,.$

Step 8: Set $u^1 \to u^0$ and return to step 1.

Observe that the red-black or odd-even nature of the basic equations means that (2.18b) and (2.19b) are explicit equations for the determination of $\hat{u}_{kj}$ and $\hat{\psi}_{ij}$ respectively.

3. Some Notation and Facts

Let $u, v \in S_h$ or S_H or S_{2h}. Then

$$(3.1) \qquad \langle u,v \rangle = \sum u_{k,j} v_{k,j}$$

where the sum is taken over all indices (k,j) so that $(x_k, y_j) \in \Omega_h$, or Ω_H or Ω_{2h} respectively. Whenever it seems that further clarity is required we will indicate the space by writing

$$\langle u,v \rangle_a , \quad a = h \text{ or } H \text{ or } 2h .$$

Since L_h, L_H and L_{2h} are positive definite operators we have the inner products

$$(3.2) \qquad [u,v]_a = \langle L_a u, v \rangle_a , \quad a = h \text{ or } H \text{ or } 2h .$$

Let

$$(3.3a) \qquad N_h := \text{Nullspace } I_h^H L_h \subset S_h$$

$$(3.3b) \qquad \mathbb{R}_h := \text{Range } I_H^h \subset S_h$$

$$(3.3c) \qquad N_H := \text{Nullspace } I_H^{2h} L_H \subset S_H$$

$$(3.3d) \qquad \mathbb{R}_H := \text{Range } I_{2h}^H \subset S_H$$

Lemma 3.1: We have

$$(3.4a) \qquad S_h = N_h \oplus \mathbb{R}_h , \quad S_H = N_H \oplus \mathbb{R}_H .$$

In fact, N_h and $\mathbb{R}_h$ are L_h orthogonal; N_H and $\mathbb{R}_H$ are L_H orthogonal. That is, if $\eta \in N_a$, $\omega \in \mathbb{R}_a$, $a = h$ or H, then

(3.4b) $$[\eta,\omega]_a = \langle L_a\eta,\omega\rangle_a = 0 \ .$$

A function $u \in S_h$ is in $\mathbb{R}_h$ if and only if

(3.5a) $$[L_h u]_{kj} = 0 \,, \quad (x_k,y_j) \in \Omega_h/\Omega_H \,.$$

A function $v \in S_h$ is in N_h if and only if

(3.5b) $$v_{kj} = 0 \,, \quad (x_k,y_j) \in \Omega_H \,.$$

A function $u \in S_H$ is in $\mathbb{R}_H$ if and only if

(3.6a) $$[L_H u]_{k,j} = 0, \quad (x_k,y_j) \in \Omega_H/\Omega_{2h} \,.$$

A function $v \in S_H$ is in N_H if and only if

(3.6b) $$v_{kj} = 0 \,, \quad (x_k,y_j) \in \Omega_{2h} \,.$$

Proof: The assertions (3.5a) and (3.6a) follow from the definition of I_h^H, I_{2h}^H etc. given by (2.9)-(2.12). The assertions (3.4a), (3.4b), (3.5b), (3.6b) now follow immediately. See [3]. ∎

Let

(3.7a) $$\hat{L}_H := I_h^H L_h I_H^h \,,$$

(3.7b) $$\hat{L}_{2h} := I_H^{2h} L_H I_{2h}^H \,.$$

Using the basic relations (2.10), (2.12) we see that

$$\text{(3.7c)} \qquad \| I_H^h v \|_{L_h}^2 = \langle L_h I_H^h v, I_H^h v \rangle_h = \frac{1}{2} \langle \hat{L}_H v, v \rangle_H ,$$

$$\text{(3.7d)} \qquad \| I_{2h}^H U \|_{L_H}^2 = \langle L_H I_{2h}^H U, I_{2h}^H U \rangle_H = \frac{1}{2} \langle \hat{L}_{2h} U, U \rangle_{2h} .$$

The formulae (2.9), (2.10), (2.11), and (2.12) together with (3.5a) and (3.6a) imply

$$\text{(3.8a)} \qquad \hat{L}_H u = \frac{1}{2} L_h I_H^h u \Big|_{\Omega_H}$$

$$\text{(3.8b)} \qquad \hat{L}_{2h} v = \frac{1}{2} L_H I_{2h}^H v \Big|_{\Omega_{2h}}$$

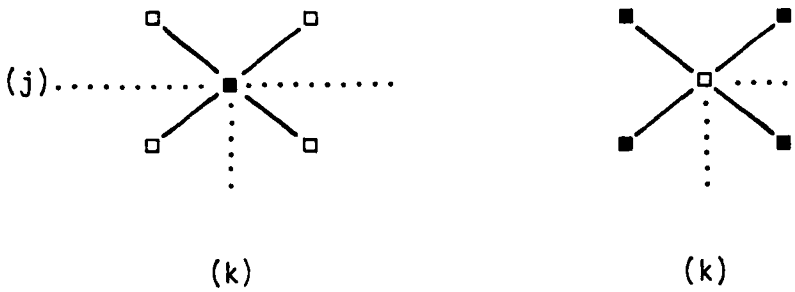
The analysis of [3] is based on the following facts about $\hat{L}_H$, $\hat{L}_{2h}$.

<u>Lemma 3.2</u>: There are operators $L_H^{(1)}$, $\tilde{L}_H^{(1)}$, $L_{2h}^{(1)}$, $\tilde{L}_{2h}^{(1)}$ such that:

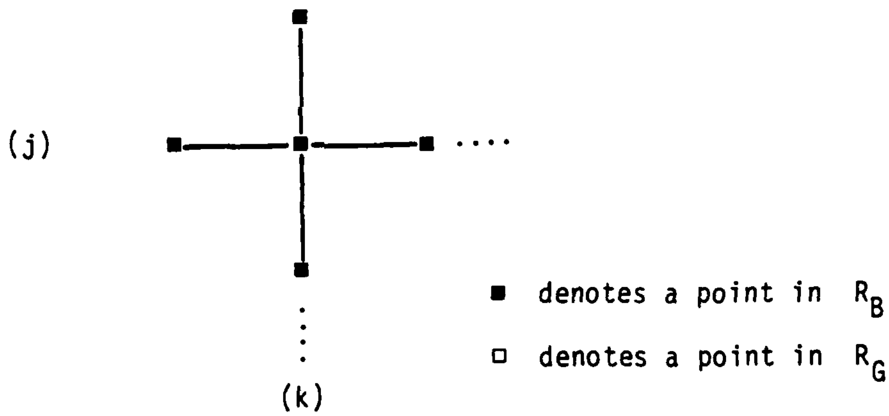
$$\text{(3.9a)} \qquad \hat{L}_H = \frac{1}{2} L_H^{(1)} + \frac{1}{2} \tilde{L}_H^{(1)} ,$$

$$\text{(3.9b)} \qquad \hat{L}_{2h} = \frac{1}{2} L_{2h}^{(1)} + \frac{1}{2} \tilde{L}_{2h}^{(1)} ,$$

The operator $L_H^{(1)}$ is based on the five points (x_k, y_j), (x_{k+1}, y_{j+1}), (x_{k-1}, y_{j+1}), (x_{k-1}, y_{j-1}), (x_{k+1}, y_{j-1}). These are the same points on which L_H is based. The operator $\tilde{L}_H^{(1)}$ is based on the five points (x_k, y_j), (x_{k+2}, y_j), (x_{k-2}, y_j), (x_k, y_{j+2}), (x_k, y_{j-2}). If $k \equiv j \equiv 0 \pmod 2$, these are the same points on which L_{2h} is based. Similarly, if $k \equiv j \equiv 0 \pmod 2$,



The five point star for $L_H, L_H^{(1)}$



The five point star for $\tilde{L}_H^{(1)}$, L_{2h}, $L_{2h}^{(1)}$

Figure 1

$L_{2h}^{(1)}$ is based on these same points. The operators $L_H^{(1)}$, $L_{2h}^{(1)}$ are "almost" the operators L_H, L_{2h}. To be precise, we have: let

(3.10a) $$a_{k-\frac12,j-\frac12} = \left[\frac{p_{k-\frac12,j}p_{k-1,j-\frac12}}{c_{k-1,j}} + \frac{p_{k,j-\frac12}p_{k-\frac12,j-1}}{c_{k,j-1}}\right],$$

(3.10b) $$b_{k+\frac12,j-\frac12} = \left[\frac{p_{k,j-\frac12}p_{k+\frac12,j-1}}{c_{k,j-1}} + \frac{p_{k+\frac12,j}p_{k+1,j-\frac12}}{c_{k+1,j}}\right],$$

(3.10c) $$d_{kj} = [a_{k-\frac12,j-\frac12} + a_{k+\frac12,j+\frac12} + b_{k+\frac12,j-\frac12} + b_{k-\frac12,j+\frac12}] .$$

If $(k+j) \equiv 0 \pmod 2$, then

(3.11) $$[L_H^{(1)}U]_{kj} = \frac{1}{h^2}\{-a_{k+\frac12,j+\frac12}U_{k+1,j+1} - a_{k-\frac12,j-\frac12}U_{k-1,j-1} - b_{k+\frac12,j-\frac12}U_{k+1,j-1} - b_{k-\frac12,j+\frac12}U_{k-1,j+1} + d_{kj}U_{kj}\} .$$

An easy computation shows that

$$|2a_{k-\frac12,j-\frac12} - p_{k-\frac12,j-\frac12}| \le Kh^2$$

$$|2b_{k+\frac12,j-\frac12} - p_{k+\frac12,j-\frac12}| \le Kh^2 .$$

Hence, for every $U \in S_H$,

(3.12a) $$|\langle L_HU,U\rangle - \langle L_H^{(1)}U,U\rangle| \le Kh^2\langle L_HU,U\rangle ,$$

(3.12b) $$|\langle L_HU,U\rangle - \langle L_H^{(1)}U,U\rangle| \le Kh^2\langle L_H^{(1)}U,U\rangle .$$

A basic estimate is: for every $U \in S_H$,

$$(3.13) \qquad 0 \le \langle \tilde{L}_H^{(1)} U, U \rangle \le 2(1+Kh) \langle L_H^{(1)} U, U \rangle .$$

Hence, if we write

$$(3.14a) \qquad \hat{L}_H = \frac{1}{2} L_H + \frac{1}{2} \tilde{L}_H^{(2)} ,$$

then

$$(3.14b) \qquad -Kh \langle L_H U, U \rangle \le \langle \tilde{L}_H^{(2)} U, U \rangle \le 2(1+Kh) \langle L_H U, U \rangle .$$

Similarly, let

$$(3.15a) \qquad A_{k+1,j} = \left[\frac{p_{k+\frac{1}{2},j+\frac{1}{2}} p_{k+\frac{3}{2},j+\frac{1}{2}}}{\bar{c}_{k+1,j+1}} + \frac{p_{k+\frac{1}{2},j-\frac{1}{2}} p_{k-\frac{3}{2},j-\frac{1}{2}}}{\bar{c}_{k+1,j-1}} \right] ,$$

$$(3.15b) \qquad B_{k,j+1} = \left[\frac{p_{k+\frac{1}{2},j+\frac{1}{2}} p_{k+\frac{1}{2},j+\frac{3}{2}}}{\bar{c}_{k+1,j+1}} + \frac{p_{k-\frac{1}{2},j+\frac{1}{2}} p_{k-\frac{1}{2},j+\frac{3}{2}}}{\bar{c}_{k-1,j+1}} \right] ,$$

$$(3.15c) \qquad D_{kj} = [A_{k+1,j} + A_{k-1,j} + B_{k,j+1} + B_{k,j-1}] .$$

If, $k \equiv j \equiv 0 \pmod 2$,

$$(3.16) \qquad L_{2h}^{(1)} = \frac{1}{2h^2} \Big\{ -A_{k+1,} U_{k+2,j} - A_{k-1,j} U_{k-2,j} - B_{k,j+1} U_{k,j+2}$$
$$- B_{k,j-1} U_{k,j-2} + D_{k,j} U_{k,j} \Big\} .$$

An easy calculation shows that

(3.17a) $$|2A_{k+1,j} - p_{k+1,j}| \le Kh^2 ,$$

(3.17b) $$|2B_{k,j+1} - p_{k,j+1}| \le Kh^2 .$$

Hence, for all $U \in S_{2h}$

(3.17c) $$|\langle L_{2h}^{(1)}U,U\rangle_{2h} - \langle L_{2h}U,U\rangle_{2h}| \le Kh^2 \langle L_{2h}U,U\rangle_{2h} .$$

The anolog of the basic estimate (3.13) holds. That is

(3.18) $$0 \le \langle \tilde{L}_{2h}^{(1)}U,U\rangle \le 2(1+Kh)\langle L_{2h}^{(1)}U,U\rangle .$$

Hence, if we write

(3.19a) $$\hat{L}_{2h} = \frac{1}{2} L_{2h} + \frac{1}{2} \tilde{L}_{2h}^{(2)}$$

then

(3.19b) $$(-Kh)\langle L_{2h}U,U\rangle \le \langle \tilde{L}_{2h}^{(2)}U,U\rangle \le 2(1+Kh)\langle L_{2h}U,U\rangle .$$

Of course, if $p(x,y) \equiv 1$, then

(3.20) $$L_H = L_H^{(1)} , \quad L_{2h} = L_{2h}^{(1)} .$$

<u>Proof</u>: The construction of $L_H^{(1)}$ and the basic estimate (3.13) is found in [3]. The construction of $L_{2h}^{(1)}$ and the estimate (3.18) then follows from the same arguments. The estimates (3.11), (3.17) are direct computations. ∎

Our next result looks at the operator $\tilde{L}_H^{(1)}$.

Lemma 3.3: The operator $\tilde{L}_H^{(1)}$ is of the form

$$(3.21) \qquad \left[\tilde{L}_H^{(1)}U\right]_{kj} = \frac{1}{h^2}\Big\{-\bar{A}_{k+1,j}U_{k+2,j} - \bar{A}_{k-1,j}U_{k-2,j} - \bar{B}_{k,j+1}U_{k,j+2} - \bar{B}_{k,j-1}U_{k,j-2} - \bar{D}_{k,j}U_{k,j}\Big\} .$$

The coefficients, $\bar{A}$, $\bar{B}$, $\bar{D}$ are given by

$$(3.22a) \qquad \bar{A}_{k+1,j} = \frac{p_{k+\frac{1}{2},j}p_{k+\frac{3}{2},j} + \frac{1}{2}(p_{k+\frac{1}{2},j})^2\theta_{k+1,j}}{c_{k+1,j}}$$

$$(3.22b) \qquad \bar{A}_{k-1,j} = \frac{p_{k-\frac{1}{2},j}p_{k-\frac{3}{2},j} + \frac{1}{2}(p_{k-\frac{1}{2},j})^2\theta_{k-1,j}}{c_{k-1,j}}$$

$$(3.22c) \qquad \bar{B}_{k,j+1} = \frac{p_{k,j+\frac{1}{2}}p_{k,j+\frac{3}{2}} + \frac{1}{2}(p_{k,j+\frac{1}{2}})^2\theta_{k,j+1}}{c_{k,j+1}}$$

$$(3.22d) \qquad \bar{B}_{k,j-1} = \frac{p_{k,j-\frac{1}{2}}p_{k,j-\frac{3}{2}} + \frac{1}{2}(p_{k,j-\frac{1}{2}})^2\theta_{k,j-1}}{c_{k,j-1}}$$

$$(3.22e) \qquad \bar{D}_{k,j} = \bar{A}_{k+1,j} + \bar{A}_{k-1,j} + \bar{B}_{k,j-1} + \bar{B}_{k,j+1}$$

where

$$\theta_{\mu,\sigma} = \begin{cases} 1, & (x_\mu, y_\sigma) \in \partial\Omega_h \\ 0, & (x_\mu, y_\sigma) \notin \partial\Omega_h \end{cases} \tag{3.23}$$

Proof: These coefficients were computed in [3]. ∎

Remark: If

$$\theta_{k\pm1,j} \neq 0 , \quad \text{then} \quad U_{k\pm2,j} = 0 ,$$

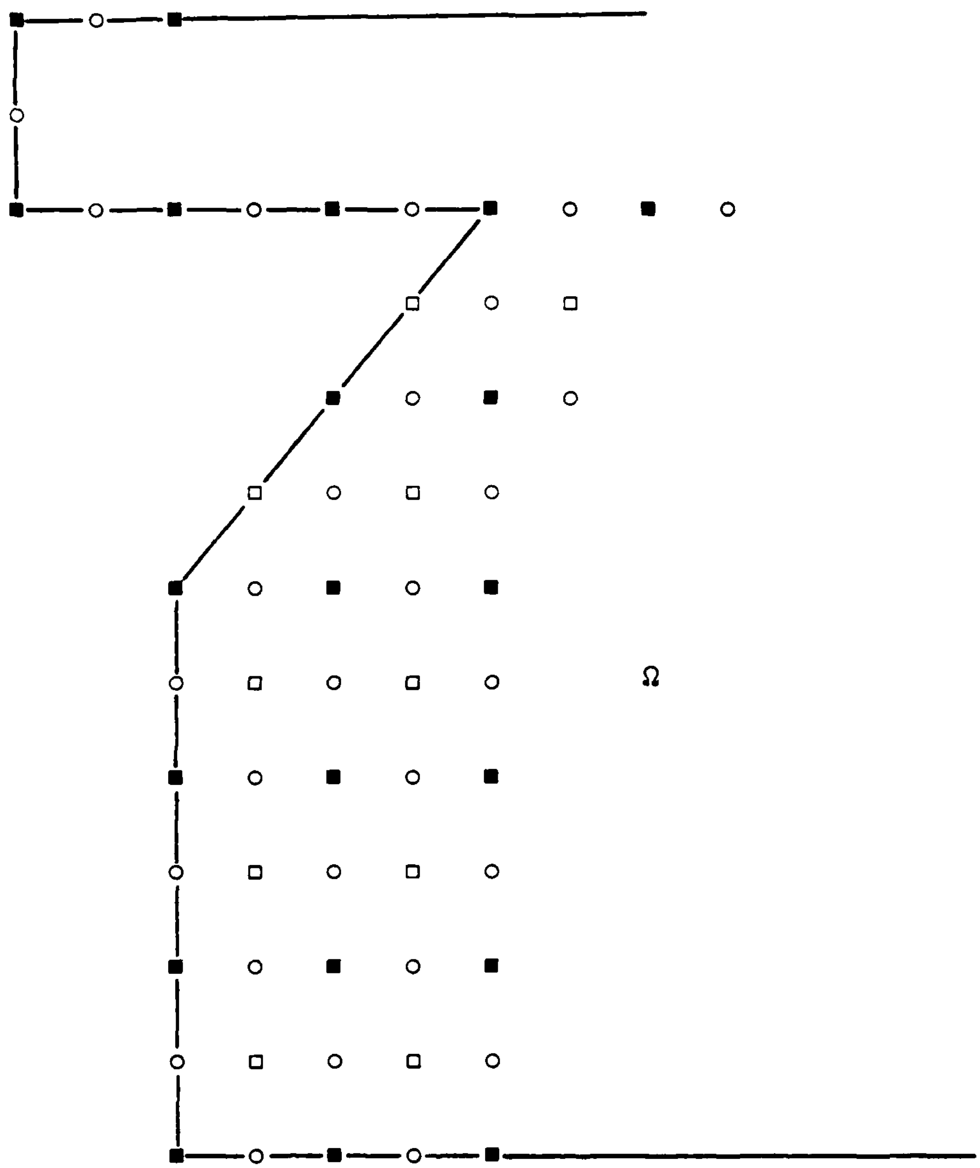
$$\theta_{k,j\pm1} \neq 0 , \quad \text{then} \quad U_{k,j\pm2} = 0 .$$

Lemma 3.4: Let $(x_k, y_j) \in \Omega_{2h}$. Then all 4 of its h grid neighbors $(x_{k\pm1}, y_j)$, $(x_k, y_{j\pm1}) \in \Omega_h$. Hence

$$\theta_{k\pm1,j} = \theta_{k,j\pm1} = 0 .$$

Proof: (See Figure 2). This result follows immediately from the fact that all corner points of $\partial\Omega$ lie in R_B. ∎

It is useful to write $\tilde{L}_H^{(1)}$ as the sum of two operators, one essentially based on Ω_{2h} and the other on Ω_H/Ω_{2h}.



Reentrant Corner

○ denotes a point in R_0

■ denotes a point in R_B

□ denotes a point in R_G

Figure 2

Definition: Let M_x, Q_x: $S_H \to S_H$ be defined by

(3.24a) $$[Q_x U]_{k,j} = 0\,, \quad (x_k, y_j) \in \Omega_H/\Omega_{2h}\,,$$

(3.24b) $$[Q_x U]_{k,j} = [\tilde{L}_H^{(1)} U]_{k,j}\,, \quad (x_k, y_j) \in \Omega_{2h}\,,$$

(3.25a) $$[M_x U]_{k,j} = [\tilde{L}_H^{(1)} U]_{k,j}\,, \quad (x_k, y_j) \in \Omega_H/\Omega_{2h}\,,$$

(3.25b) $$[M_x U]_{k,j} = 0\,, \quad (x_k, y_j) \in \Omega_{2h}\,.$$

Lemma 3.5: Let $v \in S_{2h}$. Then

(3.26) $$|\langle Q_x I_{2h}^H v, I_{2h}^H v\rangle_H - \langle L_{2h} v, v\rangle_{2h}| \le Kh^2 \langle L_{2h} v, v\rangle\,.$$

Proof: The lemma follows from Lemma 3.4 and the estimates

$$|4\bar{A}_{k+1,j} - p_{k+1,j}| \le Kh^2 p_{k+1,j}\,,$$

$$|4\bar{B}_{k,j+1} - p_{k,j+1}| \le Kh^2 p_{k,j+1}\,.$$

Remark: When $p(x,y) \equiv 1$, then $K \equiv 0$.

Finally, we "lift" L_{2h} (an operator defined on S_{2h}) as follows: let $\bar{L}_x$: $S_{2h} \to S_H$ be defined by

(3.27a) $$[\bar{L}_x(I_{2h}^H v)]_{kj} = 0\,, \quad (x_k, y_j) \in \Omega_H/\Omega_{2h}\,,$$

(3.27b) $$[\bar{L}_x(I_{2h}^H v)]_{k,j} = [L_{2h} v]_{k,j}\,, \quad (x_h, y_j) \in \Omega_{2h}\,.$$

Remark: Using this definition we may rephrase (3.26) as

(3.28) $$|\langle Q_x I_{2h}^H v, I_{2h}^H v\rangle_H - \langle \bar{L}_x I_{2h}^H v, I_{2h}^H v\rangle_H| \le Kh^2 \langle \bar{L}_x I_{2h}^H v, I_{2h}^H v\rangle_H\,.$$

4. Analysis I.

Let $\varepsilon^0 = u - u^0$ be the initial error. Then $\tilde{\varepsilon} = u - \tilde{u}$ is the error after step 1, the smoothing step. Assumption (2.16) asserts that

$$(4.1) \qquad \|\tilde{\varepsilon}\|^2_{L_h} = \langle L_h\tilde{\varepsilon},\tilde{\varepsilon}\rangle \le \langle L_h\varepsilon^0,\varepsilon^0\rangle = \|\varepsilon^0\|^2_{L_h} .$$

Using the decomposition (3.4a) we have

$$(4.2) \qquad \tilde{\varepsilon} = \eta_h + I^h_H w , \quad \eta_h \in N_h , \quad w \in S_H .$$

From step 2 [i.e. (2.18)] of the algorithm and Lemma 3.1 [i.e. (3.5b)] we see that

$$(4.3) \qquad \hat{\varepsilon} = u - \hat{u} = I^h_H w .$$

Hence, using (3.4a) we see that

$$(4.4) \qquad \|\hat{\varepsilon}\|^2_{L_h} = \|I^h_H w\|^2_{L_h} \le \|\eta_h\|^2_{L_h} + \|I^h_H w\|^2_{L_h} = \|\tilde{\varepsilon}\|^2_{L_h} \le \|\varepsilon^0\|^2_{L_h} .$$

Using (4.3) and (3.7a) and step 3 of the algorithm we see that

$$(4.5) \qquad \hat{L}_H w = (I^H_h L_h I^h_H) w = r_H .$$

See [3] for a more complete discussion of the significance of this fact.

Lemma 4.1: Let $v \in S_H$ be the solution of

$$(4.6) \qquad L_H v = r_H = \hat{L}_H w .$$

Let

$$(4.7) \qquad v = \eta_H + I^H_{2h} V , \quad \eta_H \in N_H , \quad V \in S_{2h} .$$

Let $\hat{\psi}$ be the function in S_H constructed in step 4 [i.e. (2.19)] of the algorithm. Then

$$\hat{\psi} = \eta_H . \tag{4.8}$$

Proof: Observe that (2.19a) and (3.5b) imply that $\hat{\psi} \in N_H$. Also (2.19b) and (4.6) yield

$$[L_H(v-\hat{\psi})]_{k,j} = 0 , \quad (x_k, y_j) \in \Omega_H/\Omega_{2h} .$$

That is

$$(v-\hat{\psi}) = [(\eta_H-\hat{\psi}) + I^N_{2h}V] \in \mathbb{R}_H \tag{4.9a}$$

while

$$(\eta_H-\hat{\psi}) \in N_H . \tag{4.9b}$$

Using (3.4a) and (3.4b) we see that (4.8) holds. ∎

Consider the function ϕ which is constructed in step 6 of the algorithm. We have

$$L_{2h}\phi = I^{2h}_H L_H(v-\hat{\psi}) = I^{2h}_H L_H I^H_{2h} V . \tag{4.10}$$

thus

$$L_{2h}\phi = \hat{L}_{2h} V . \tag{4.11}$$

From (4.3), (4.11) and step 7 of the algorithm we see that

$$(4.12)\qquad \varepsilon^1 = u - u^1 = I_H^h[(w-\hat{\psi}) - I_{2h}^H\phi] \in \mathbb{R}_h .$$

Thus, if we seek an eigenfunction ε^0, it must have the form

$$\varepsilon^0 = I_H^h \varepsilon_H .$$

As we shall see, the generality of G_h and the estimate (4.1) implies that it suffices to consider the case where $G_h = I_h$. In that case

$$(4.13)\qquad \tilde{\varepsilon} = \varepsilon^0 = I_H^h[\bar{\eta}_H + I_{2h}^H U] ; \quad \bar{\eta}_H \in N_H , \quad U \in S_{2h} .$$

If $\varepsilon^1 = \mu\varepsilon^0$ (4.12) becomes

$$\varepsilon^1 = I_H^h[(\bar{\eta}_H - \hat{\psi}) + I_{2h}^H(U-\phi)] = \mu I_H^h[\bar{\eta}_H + I_{2h}^H U] .$$

Thus

$$(4.14)\qquad \hat{\psi} = \lambda\bar{\eta}_H , \quad \phi = \lambda U , \quad \lambda = (1-\mu) .$$

Returning to Lemma 4.1 we have

$$(4.15)\qquad \begin{aligned} L_H(\hat{\psi} + I_{2h}^H V) &= \hat{L}_H(\bar{\eta}_H + I_{2h}^H U) \\ L_H(\lambda\bar{\eta}_H + I_{2h}^H V) &= \hat{L}_H(\bar{\eta}_H + I_{2h}^H U) . \end{aligned}$$

From (3.8b), (3.6a), (4.11) and (4.14) we see that

$$(4.16a)\qquad L_H I_{2h}^H V\Big|_{\Omega_{2h}} = 2\hat{L}_{2h} V = 2L_h\phi = 2\lambda L_{2h} U$$

$$(4.16b)\qquad L_H I_{2h}^H V\Big|_{\Omega_H/\Omega_{2h}} = 0 .$$

Thus, (4.16) and the definition of $\bar{L}_x$[i.e. (3.27)] allows us to rewrite (4.15) as

(4.17) $$\lambda[L_H\bar{\eta}_H + 2\bar{L}_x I_{2h}^H U] = \hat{L}_H[\bar{\eta}_H + I_{2h}^H U] .$$

To simplify the eigenvalue problem (4.17) we define

$$L^{\#}: S_H \to S_H$$

as follows: let $v \in S_H$. Then there is a unique representation

(4.18a) $$v = \zeta_H + I_{2h}^H W , \quad \zeta_H \in N_H , \quad W \in S_{2h} .$$

Then

(4.18b) $$L^{\#}v = L_H\zeta_H + 2\bar{L}_x I_{2h}^H W .$$

The eigenvalue problem (4.18) now becomes

(4.19a) $$\lambda L^{\#}v = \hat{L}_H v ,$$

(4.19b) $$v = \bar{\eta}_H + I_{2h}^H U .$$

Observe that both $L^{\#}$ and $\hat{L}_H$ are symmetric positive definite operators. Therefore, there is a complete set of eigenfunctions $\{v_k\}$ which satisfy

(4.20) $$\langle L^{\#}v_k, v_j \rangle = \langle \hat{L}_H v_k, v_j \rangle = 0 , \quad k \neq j .$$

Then (3.7c) implies that

(4.21) $$\frac{\| \varepsilon^1 \|_{L_h}}{\| \varepsilon^0 \|_{L_h}} \leq \max |1-\lambda| = \max |\mu| .$$

Thus, in view of (4.1), the general three-grid iteration $(G_h \neq I_h)$ also satisfies (4.21).

5. Analysis II

Consider the basic eigenvalue problem (4.19). Let us now focus our attention on the left-hand-side of (4.19a). Using (3.9a) and (4.19b) we have

$$\hat{L}_H v = \frac{1}{2} L_H^{(1)}\bar{\eta}_H + \frac{1}{2} L_H^{(1)} I_{2h}^H U + \frac{1}{2} \tilde{L}_H^{(1)}\bar{\eta}_H + \frac{1}{2} \tilde{L}_H^{(1)} I_{2h}^H U, \tag{5.1a}$$

and

$$\begin{aligned} \langle v,\hat{L}_H v\rangle = {} & \frac{1}{2}\langle \bar{\eta}_H, L_H^{(1)}\bar{\eta}_H\rangle + \frac{1}{2}\langle \bar{\eta}_H, L_H^{(1)} I_{2h}^H U\rangle + \frac{1}{2}\langle \bar{\eta}_H, \tilde{L}_H^{(1)}\bar{\eta}_H\rangle \\ & + \frac{1}{2}\langle \bar{\eta}_H, \tilde{L}_H^{(1)} I_{2h}^H U\rangle + \frac{1}{2}\langle I_{2h}^H U, L_H^{(1)}\bar{\eta}_H\rangle + \frac{1}{2}\langle I_{2h}^H U, L_H^{(1)} I_{2h}^H U\rangle \\ & + \frac{1}{2}\langle I_{2h}^H U, \tilde{L}_H^{(1)}\bar{\eta}_H\rangle + \frac{1}{2}\langle I_{2h}^H U, \tilde{L}_H^{(1)} I_{2h}^H U\rangle . \end{aligned} \tag{5.1b}$$

The basic estimate (3.12a) allows us to replace $L_H^{(1)}$ by L_H provided we accept error terms of the form

$$\delta_1 = Kh^2[\langle L_H I_{2h}^H U, I_{2h} U\rangle \langle L_H\bar{\eta}_H, \bar{\eta}_H\rangle]^{1/2}, \tag{5.2a}$$

$$\delta_2 = Kh^2\langle L_H I_{2h}^H U, I_{2h} U\rangle, \tag{5.2b}$$

$$\delta_3 = Kh^2\langle L_H\bar{\eta}_H, \bar{\eta}_H\rangle . \tag{5.2c}$$

Thus we may rewrite (5.1b) as

$$\begin{aligned} \langle v,\hat{L}_H v\rangle = {} & \frac{1}{2}\langle \bar{\eta}_H, L_H\bar{\eta}_H\rangle + \frac{1}{2}\langle I_{2h}^H U, L_H I_{2h}^H U\rangle \\ & + \frac{1}{2}\langle \bar{\eta}_H, \tilde{L}_H^{(1)}\bar{\eta}_H\rangle + \langle \bar{\eta}_H, \tilde{L}_H^{(1)} I_{2h}^H U\rangle + \frac{1}{2}\langle I_{2h}^H U, \tilde{L}_H^{(1)} I_{2h}^H U\rangle + O(\delta) \end{aligned} \tag{5.3}$$

where

(5.4) $$O(\delta) = O(\delta_1 + \delta_2 + \delta_3) .$$

From (3.6b) of Lemma (3.1) we see that

$$(\bar{\eta}_H)_{k,j} = 0 , \quad (x_k, y_j) \in \Omega_{2h} .$$

Hence

(5.5a) $$\langle \bar{\eta}_H, \tilde{L}_H^{(1)} \bar{\eta}_H \rangle = \langle \bar{\eta}_H, M_x \bar{\eta}_H \rangle ,$$

(5.5b) $$\langle \bar{\eta}_H, \tilde{L}_H^{(1)} I_{2h}^H U \rangle = \langle \bar{\eta}_H, M_x I_{2h}^H U \rangle .$$

Thus, we may rewrite (5.3) as

$$\langle v, \hat{L}_H v \rangle = \frac{1}{2} \langle \bar{\eta}_H, L_H \bar{\eta}_H \rangle + \frac{1}{2} \langle I_{2h}^H U, L_H I_{2h}^H U \rangle + \frac{1}{2} \langle \bar{\eta}_H, M_x \bar{\eta}_H \rangle$$

(5.6) $$+ \langle \bar{\eta}_H, M_x I_{2h}^H U \rangle + \frac{1}{2} \langle I_{2h}^H U, M_x I_{2h}^H U \rangle +$$

$$\frac{1}{2} \langle I_{2h}^H U, Q_x I_{2h}^H U \rangle + O(\delta) .$$

Let us consider the term

(5.7a) $$J := \frac{1}{2} \langle I_{2h}^H U, L_H I_{2h}^H U \rangle_H .$$

From (3.7b), (2.12) and (3.9b), (3.19b) we have

(5.7b) $$J = \langle U, \hat{L}_{2h} U \rangle_{2h} = \frac{1}{2} \langle U, L_{2h} U \rangle_{2h} + \frac{1}{2} \langle U, \tilde{L}_{2h}^{(2)} U \rangle_{2h} .$$

Thus, using the definition of $\bar{L}_x$ and (3.17c) we obtain

$$(5.7c) \qquad J = \frac{1}{2}\langle I_{2h}^{H}U, \bar{L}_x I_{2h}^{H}U\rangle_H + \frac{1}{2}\langle U, \tilde{L}_{2h}^{(2)}U\rangle_{2h} .$$

The estimate (3.28) allows us to replace Q_x by $\bar{L}_x$ provided we accept errors of the form

$$(5.8) \qquad \bar{\delta} = Kh^2 \langle \bar{L}_x I_{2h}^{H}U, I_{2h}^{H}U\rangle = Kh^2 \langle L_H I_{2h}^{H}U, I_{2h}^{H}U\rangle .$$

Thus, we rewrite (5.6) as

$$(5.9) \qquad \begin{aligned} \langle v, \hat{L}_H v\rangle = {} & \frac{1}{2}\langle v, L^{\#}v\rangle + \frac{1}{2}\langle U, \tilde{L}_{2h}^{(2)}U\rangle_{2h} \\ & + \frac{1}{2}\langle v, M_x v\rangle + O(\delta) + O(\bar{\delta}) . \end{aligned}$$

The eigenvalue problem (4.19) becomes

$$(5.10) \qquad (\lambda-\tfrac{1}{2})\langle v, L^{\#}v\rangle = \frac{1}{2}\langle U, \tilde{L}_{2h}^{(2)}U\rangle_{2h} + \frac{1}{2}\langle v, M_x v\rangle + O(\delta+\bar{\delta}) .$$

Hence

$$\lambda - \frac{1}{2} \geq -Kh^2$$

and

$$(5.11) \qquad \lambda \geq \frac{1}{2}(1-Kh^2) .$$

The complete proof of our basic estimate requires a more detailed analysis of the terms which appear in (5.10).

<u>Lemma 5.1</u>: For all $\eta \in N_H$ we have

$$0 \leq \langle M_x\eta,\eta\rangle \leq (1+Kh)\,\langle L_H\eta,\eta\rangle \;. \tag{5.12}$$

<u>Proof</u>: Using the description of $\tilde{L}_H^{(1)}$ given in Lemma 3.3 we see that

$$\langle M_x\eta,\eta\rangle \leq \frac{2}{h^2}\sum p_{kj}(1+Kh)(\eta_{kj})^2 \;. \tag{5.13}$$

On the other hand, since $\eta_{kj} = 0$ if $(x_k,y_j) \in \Omega_{2h}$, (2.13b) shows that

$$\langle L_H\eta,\eta\rangle = \frac{1}{2h^2}\sum \bar{c}_{kj}(\eta_{kj})^2 \geq \frac{2}{h^2}\sum p_{kj}(1-Kh^2)(\eta_{kj})^2 \;. \tag{5.14}$$

Thus, the lemma is proven.

Our next result is intuitively clear. Nevertheless, the details are somewhat technical.

<u>Lemma 5.2</u>: For every $U \in S_{2h}$, we have

$$\langle I_{2h}^H U, M_x I_{2h}^H U\rangle \leq (1+Kh)\,\langle I_{2h}^H U, \bar{L}_x I_{2h}^H U\rangle \;. \tag{5.15}$$

<u>Proof</u>: The idea behind the proof is quite simple. We have

$$[\bar{L}_x I_{2h}^H U]_{kj} = [L_{2h}U]_{k,j} \qquad (x_k,y_j) \in \Omega_{2h}$$

while [see (2.11)] $\{(I_{2h}^H U)_{k,j}\}$ for $(x_k,y_j) \in \Omega_H/\Omega_{2h}$ is an "average" of $\{U_{\sigma\mu}\}$ with $(x_\sigma,y_\mu) \in \Omega_{2h}$. Since M_x and $\bar{L}_x$ are "almost" the same operator, (5.15) should follow. The complete details of the proof are given in the Appendix. ∎

<u>Theorem 5.1</u>: Consider the three grid iterative scheme described in section 2: steps 1-8. Let

$$\varepsilon^0 = u - u^0 , \quad \varepsilon^1 = u - u^1 .$$

There is a constant $K \geq 0$, depending only on $p(x,y)$ and its first and second derivatives, such that

$$\|\varepsilon^1\|_{L_h} \leq \frac{1}{2}(1+Kh)\|\varepsilon^0\|_{L_h} . \tag{5.16}$$

Moreover, if $p(x,y) \equiv \text{const} > 0$, then $K = 0$.

<u>Proof</u>: Let (λ,v) be an eigenvalue and eigenfunction of (4.19), or equivalently, (4.17). As we have seen, (λ,v) satisfy (5.10) and (5.11) holds. Expanding the terms of $\langle v,M_x v\rangle$ we have

$$\text{R.H.S} = \frac{1}{2}\langle U,\tilde{L}_{2h}^{(2)}U\rangle_{2h} + \frac{1}{2}[\langle\bar{\eta}_H,M_x\bar{\eta}_H\rangle + 2\langle\bar{\eta}_H,M_x I_{2h}^H U\rangle + \langle I_{2h}^H U,M_x I_{2h}^H U\rangle] + O(\delta+\bar{\delta}) . \tag{5.17}$$

Using (3.19b), (3.27), Lemma 5.1 and Lemma 5.2 we have

$$\begin{aligned}\text{R.H.S} &\leq (1+Kh)\langle I_{2h}^H U,\bar{L}_x I_{2h}^H U\rangle + \langle\bar{\eta}_H,M_x\bar{\eta}_H\rangle + \langle I_{2h}^H U,M_x I_{2h}^H U\rangle + O(\delta+\bar{\delta}) \\ &\leq (1+Kh)[2\langle I_{2h}^H U,\bar{L}_x I_{2h}^H U\rangle + \langle\bar{\eta}_H,L_H\bar{\eta}_H\rangle] + O(\delta+\bar{\delta}) .\end{aligned}$$

Thus

$$(\lambda - 3/2)\ \langle v, L^{\#}v \rangle \le O(\delta+\bar{\delta}) + Kh\ \langle v, L^{\#}v \rangle\ .$$

Hence

(5.18) $$\lambda \le \frac{3}{2}\,(1+Kh)\ .$$

Thus, (5.18) and (5.11) together with the remarks at the end of section 4 imply the theorem.

Appendix

In this appendix we give the details of the proof of Lemma 5.2. We use the formulae of Lemma 3.3 which give the form of $\tilde{L}_H^{(1)}$ and hence give the form of M_x. A simple summation-by-parts argument shows that

$$\text{(A.1)} \qquad \langle I_{2h}^H U, M_x I_{2h}^H U \rangle = \frac{1}{h^2} \sum_{R_G} \bar{A}_{k+1,j}[(I_{2h}^H U)_{k+2,j} - (I_{2h}^H U)_{k,j}]^2$$

$$+ \frac{1}{h^2} \sum_{R_G} \bar{B}_{k,j+1}[(I_{2h}^H U)_{k,j+2} - (I_{2h}^H U)_{k,j}]^2 ,$$

while

$$\text{(A.2)} \qquad \langle I_{2h}^H U, \bar{L}_x I_{2h}^H U \rangle = \frac{1}{4h^2} \sum_{R_B} p_{k+1,j}[U_{k+2,j} - U_{k,j}]^2$$

$$+ \frac{1}{4h^2} \sum_{R_B} p_{k,j+1}[U_{k,j+2} - U_{k,j}]^2 .$$

<u>Note</u>: It is essential to observe that the sums in (A.1) are taken over points $(x_k, y_j) \in R_G$ while the sums in (A.2) are taken over points in R_B. Moreover, if $(x_k, y_j) \in R_B$ then

$$(I_{2h}^H U)_{kj} = U_{k,j} .$$

Let $j \equiv 1 \pmod 2$ be fixed and consider the contribution to the first sum (on the right-hand-side) of (A.1) from the points on a connected segment of the intersection of Ω with the line $y = jh$. That is, we consider a sum

$$(A.3) \qquad \sum_j = \frac{1}{h^2} \sum_{r=r_0}^{r_1} \bar{A}_{2r,j}[(I_{2h}^H U)_{2r+1,j} - (I_{2h}^H U)_{2r-1,j}]^2 .$$

For $r_0 \le r \le r_1 - 1$ all the points $(x_{2r+1}, y_j) \in \Omega$. The points (x_{2r_0-1}, y_j), (x_{2r_1+1}, y_j) may lie on $\partial\Omega$ or may lie outside $\bar{\Omega}$(*) depending on the slope of the boundary near these points. In either case $(I_{2h}^H U)_{2r_0-1,j} = (I_{2h}^H U)_{2r_1+1,j} = 0$.

Consider the contribution of such an "end-point" to the sum $\sum_j$ (See Fig. 3). For definiteness, consider the term

$$(A.4) \qquad J_{2r_0,j} = A_{2r_0,j}[(I_{2h}^H U)_{2r_0+1,j} - (I_{2h}^H U)_{2r_0-1,j}]^2 .$$

We should consider two cases, either the boundary has slope ∞ near (x_{2r_0}, y_j) or slope ± 1. When the slope is ± 1, then $\theta_{2r_0,j} = 0$ and the analysis is much like the case when both (x_{2r+1}, y_j), $(x_{2r-1}, y_j) \in \Omega$. However, when the slope is ∞, then $\theta_{2r_0,j} \neq 0$ (see Fig. 2). In this case

$$(I_{2h}^H U)_{2r_0-1,j} = 0 ,$$

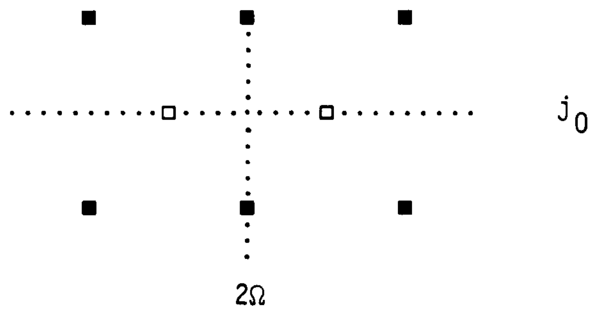
$$|(I_{2h}^H U)_{2r_0+1,j}| \le \frac{(1+Kh)}{4} [|U_{2r_0+2,j+1}| + |U_{2r_0+2,j-1}|] ,$$

and

$$0 \le A_{2r_0,j} \le \frac{3}{8} (1+Kh) p_{2r_0+1,j\pm 1} .$$

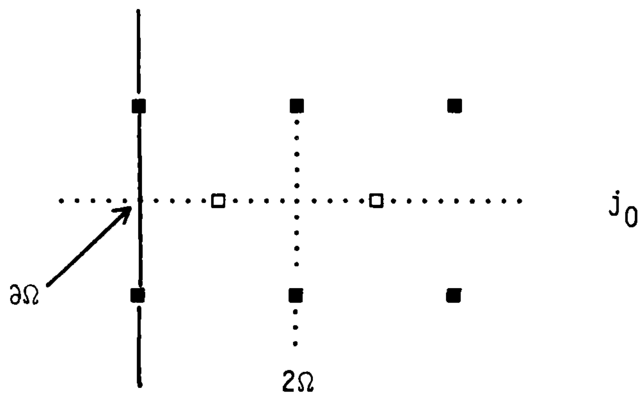
---

*Near a reentrant corner with angle 45° one of these points may actually lie on $\partial\Omega$ while the segment from its neighbor, say (x_{2r_0+1}, y_j), is not entirely in $\bar{\Omega}$. Nevertheless, $U = 0$ at that point.



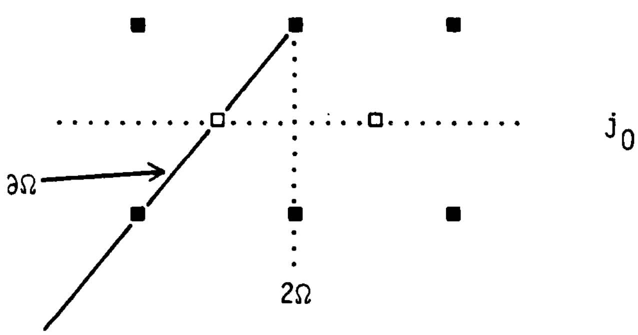
(a)

Totally interior



(b)

Near the boundary



(c)

Near an oblique boundary edge

■ denotes a point in R_B

□ denotes a point in R_G

Figure 3

Hence

$$J_{2r_0,j} \le \frac{3}{16} \cdot \frac{1}{4}(1+Kh)\Big\{p_{2r_0+1,j+1}[U_{2r_0+2,j+1} - U_{2r_0,j+1}]^2$$

(A.5)

$$+ p_{2r_0+2,j-1}[U_{2r_0+2,j-1} - U_{2r_0,j-1}]^2\Big\} .$$

Now, consider a term

(A.6) $$J_{2r,j} = A_{2r,j}[(I^H_{2h}U)_{2r+1,j} - (I^H_{2h}U)_{2r-1,j}]^2$$

where $r_0 < r < r_1$. We write

$$I_{r,j} = (I^H_{2h}U)_{2r+1,j} - (I^H_{2h}U)_{2r-1,j}$$

as a sum of four terms

$$I_{r,j} = D_{1,j+1} + D_{2,j+1} + D_{1,j-1} + D_{2,j-1}$$

where - up to terms of order h^2

(A.7a) $$D_{1,j+1} = \frac{p_{2r+3/2,j+1/2}}{4p_{2r+1,j}} U_{2r+2,j+1} - \frac{p_{2r-1/2,j+1/2}}{4p_{2r-1,j}} U_{2r,j+1} ,$$

(A.7b) $$D_{2,j+1} = \frac{p_{2r+1/2,j+1/2}}{4p_{2r+1,j}} U_{2r,j+1} - \frac{p_{2r-3/2,j+1/2}}{4p_{2r-1,j}} U_{2r-2,j+1} .$$

Indeed, expanding the coefficients about (x_{2r},y_j) yields

$$|I_{r,j}| \leq \frac{1+Kh}{4} \{|U_{2r+2,j+1} - U_{2r,j+1}| + |U_{2r,j+1} - U_{2r-2,j+1}|$$

$$+ |U_{2r+2,j-1} - U_{2r,j-1}| + |U_{2r,j-1} - U_{2r-2,j-1}|\} .$$

Hence

$$J_{2r,j} \leq \frac{1+Kh}{16} \{p_{2r+1,j+1}[U_{2r+2,j+1} - U_{2r,j+1}]^2 +$$

$$p_{2r-1,j+1}[U_{2r,j+1} - U_{2r-2,j+1}]^2 + p_{2r+1,j-1}[U_{2r+2,j-1} - U_{2r,j-1}]^2 +$$

$$p_{2r-1,j-1}[U_{2r,j-1} - U_{2r-2,j-1}]^2\} .$$

Upon adding the contribution from each j line, we see that each term

$$\frac{1}{4h^2} p_{2r+1,j+1}[U_{2r+2,j+1} - U_{2r,j+1}]^2$$

enters at most 4 times. Since each such term has a coefficient which is less than or equal to $(1+Kh)/4$, the lemma is proven. ∎

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